Solution to HW8 for MATH 2020A (2024 Fall)

1. (Q6)

$$\operatorname{curl}(F)\cdot \hat{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{y}{x}\right) - \frac{\partial}{\partial y}\left(\frac{x}{y}\right)$$

$$= \frac{y}{x^2} + \frac{x}{y^2}$$

$\blacksquare$

2. (Q10) Verify $\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$ ①

and $\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right)dx\,dy.$ ②

$\vec{F}(x,y) = (\underset{M}{-x^2y}, \underset{N}{xy^2})$, $C: \vec{r}(t) = (a\cos t, a\sin t)$, $0\le t\le 2\pi$.

$\vec{r}'(t) = (-a\sin t, a\cos t)$

So LHS of ① $= \int_C \vec{F}\cdot\vec{r}'\,ds = \int_0^{2\pi} (-a^2\cos^2 t\cdot a\sin t)\cdot(-a\sin t) + (a\cos t\cdot a^2\sin^2 t)\cdot a\cos t\,dt$

$= \int_0^{2\pi} \left(a^4\cos^2 t\sin^2 t + a^4\sin^2 t\cos^2 t\right)dt$

$= 2a^4\int_0^{2\pi} \frac{1}{4}\sin^2 2t\,dt$

$= \frac{1}{2}a^4\int_0^{2\pi}\frac{1-\cos 4t}{2}\,dt = \frac{1}{2}\pi a^4.$

RHS of ② $= \iint_R (M_x+N_y)\,dA$

$= \iint_R (-2xy+2xy)\,dA = 0$

$=$ LHS of ②.

RHS of ① $= \iint_R y^2-(-x^2)\,dA = \int_0^{2\pi}\int_0^a \rho^2\cdot\rho\,d\rho\,d\theta$

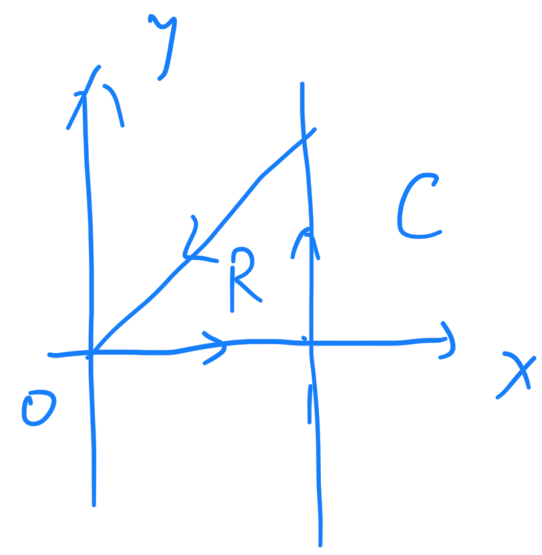
$= \frac{1}{2}\pi a^4 =$ LHS of ①.

LHS of ② $= \int_C -N\,dx + M\,dy = \int_0^{2\pi} -a\cos t\cdot a^2\sin^2 t\,(-a\sin t)\,dt + -a^2\cos^2 t\cdot a\sin t\cdot a\cos t\,dt$

$= a^4\int_0^{2\pi}\sin t\cos t\,(\sin^2 t-\cos^2 t)\,dt$

$= -a^4\int_0^{2\pi}\frac{1}{2}\sin 2t\cdot\cos 2t\,dt = -\frac{1}{4}a^4\int_0^{2\pi}\sin 4t\,dt = 0$

3. (Q14)



$$\int_C \vec{F}\cdot\vec{T}\,ds = \iint_R (N_x - M_y)\,dA = \iint_R (-2x-1)\,dA$$

$$= \int_0^1 \int_0^x (-2x-1)\,dy\,dx = \int_0^1 (-2x^2 - x)\,dx$$

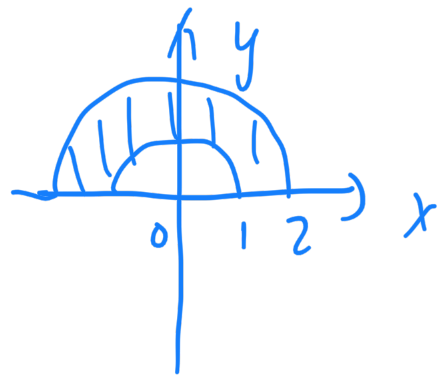
$$= -\frac{2}{3} - \frac{1}{2} = -\frac{7}{6}.$$

$$\int_C \vec{F}\cdot\vec{n}\,ds = \iint_R (M_x + N_y)\,dA = \iint_R (1-2y)\,dA$$

$$= \int_0^1 \int_0^x (1-2y)\,dy\,dx$$

$$= \int_0^1 (x - x^2)\,dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

$\boxtimes$



4. (Q20)

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R (N_x - M_y)\,dA = \iint_R \left(\frac{2x}{x^2+y^2} - \frac{x}{x^2+y^2}\right) dA$$

$$= \int_0^{\pi}\int_1^2 \frac{\rho\cos\theta}{\rho^2}\,\rho\,d\rho\,d\theta$$

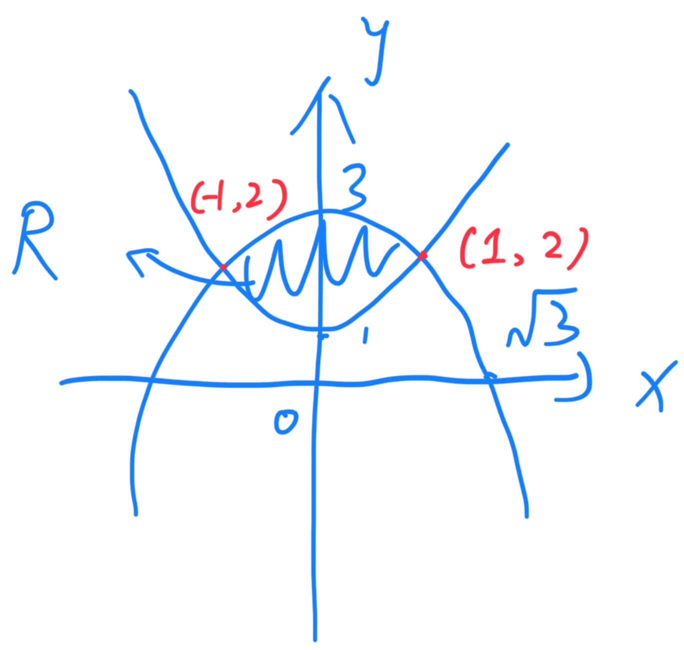
$$= \int_0^{\pi}\int_1^2 \cos\theta\,d\rho\,d\theta = 0$$

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R (M_x + N_y)\,dA = \iint_R \frac{-y}{x^2+y^2} + \frac{2y}{x^2+y^2}\,dA$$

$$= \int_0^{\pi}\int_1^2 \frac{\rho\sin\theta}{\rho^2}\,\rho\,d\rho\,d\theta$$

$$= \int_0^{\pi}\int_1^2 \sin\theta\,d\rho\,d\theta = 2.$$

5. (Q24)



$$\int_C \vec{F}\cdot\vec{T}\,ds = \iint_R (N_x - M_y)\,dA = \iint_R (-1)\,dA$$

$$= -\int_{-1}^{1}\int_{x^4+1}^{3-x^2} 1\,dy\,dx$$

$$= -\int_{-1}^{1} 2-x^2-x^4\,dx$$

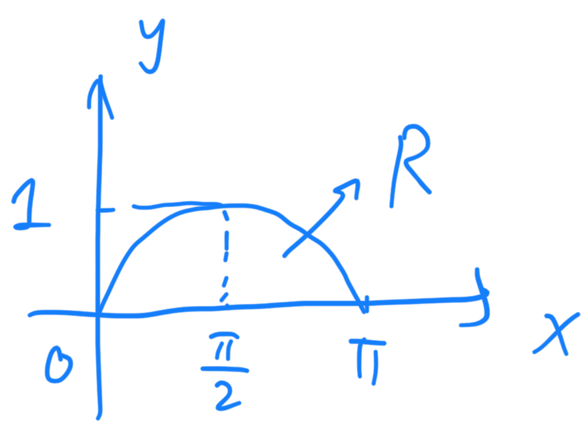
$$= -2\int_0^1 2-x^2-x^4\,dx$$

$$= -2\left(2-\frac{1}{3}-\frac{1}{5}\right)$$

$$= -\frac{44}{15}.$$

$\boxtimes$

6. (Q28)



$$\oint_C 3y\,dx + 2x\,dy = \iint_R \left(\frac{\partial (2x)}{\partial x} - \frac{\partial (3y)}{\partial y} \right) dA$$

$$= -\iint_R 1\,dA = -\int_0^{\pi} \int_0^{\sin x} dy\,dx$$

$$= -\int_0^{\pi} \sin x\,dx = -2 .$$

$\boxtimes$

7. (Q32)

$$A(R) = \frac{1}{2} \oint_C x\,dy - y\,dx$$

$$= \frac{1}{2} \int_0^{2\pi} a\cos t \; b\cos t \, dt + b\sin t \; a\sin t \, dt$$

$$= \frac{1}{2} ab \int_0^{2\pi} dt = ab\pi .$$

$\boxtimes$

8. (Q36) Since R is a square, we may apply Green's thm :

simple domain with closed piecewise smooth boundary

$$\oint_C xy^2\,dx + (x^2y+2x)\,dy$$

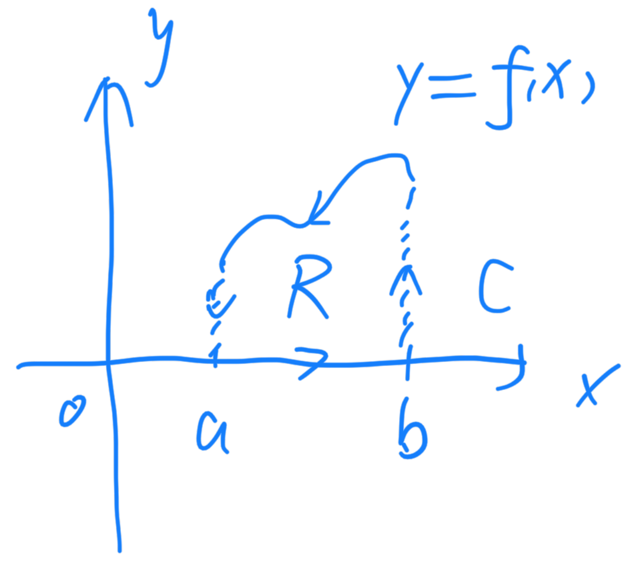
$$\overset{\text{Green's thm}}{=} \iint_R \frac{\partial}{\partial x}(x^2y+2x) - \frac{\partial}{\partial y}(xy^2)\,dA$$

$$= \iint_R 2xy + 2 - 2xy \;dA$$

$= 2A(R)$, which depends only on the area of the square and not on its location

$\square$

9. (Q40)



Although R may not be the union of finitely many simple domains, Green's thm. still holds for this type of domain (we omit the proof).

So $\oint_C y\,dx = \iint_R -1\,dA = -A(R)$,

and $A(R) = \int_a^b |f(x)|\,dx = \int_a^b f(x)\,dx$ because $f \geq 0$ on $[a,b]$.

Hence $\int_a^b f(x)dx = -\oint_C y\,dx$. ▨

Q45. (a) $C: \vec{r}(t) = (a\cos t, a\sin t)$, $0 \leq t \leq 2\pi$.

$f_x = \frac{2x}{x^2+y^2}$, $f_y = \frac{2y}{x^2+y^2}$, $\vec{n} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$.

$$\oint_C \nabla f \cdot \vec{n}\,ds = \oint_C \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}\right) \cdot \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right) ds$$

$$= \int_C \frac{2}{\sqrt{x^2+y^2}}\,ds = \int_0^{2\pi} \frac{2}{\rho} \cdot \sqrt{\rho^2\cos^2\theta + \rho^2\sin^2\theta}\,d\theta = 4\pi.$$

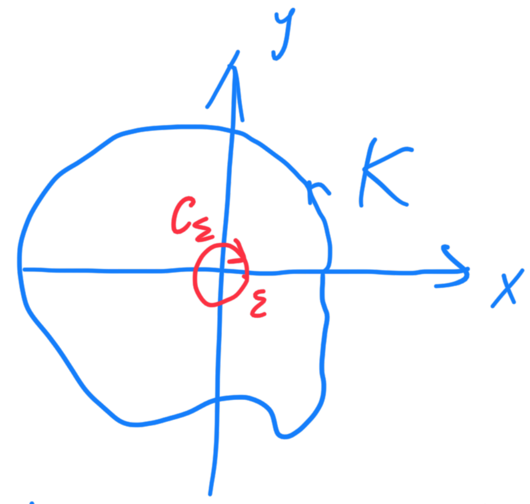
(b) Let R be the domain enclosed by K.

Case 1: $(0,0) \in R$,

then $\nabla f \notin C^1(\bar{R})$ because $\vec{0}$ is the singularity.

But $\nabla f \in C^1(\overline{R_\varepsilon})$:

10. (Q45)



Apply Green's thm to the domain R_ε enclosed by K & C_ε: $x^2+y^2=\varepsilon^2$ (clockwise orientation):

$$\oint_{K \sqcup C_\varepsilon} \nabla f \cdot \vec{n}\, ds = \int_{R_\varepsilon} \nabla \cdot \nabla f \, dA = \int_{R_\varepsilon} \Delta f \, dA.$$

$$\Delta f = f_{xx} + f_{yy}, \quad f_x = \frac{2x}{x^2+y^2}, \quad f_{xx} = \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2}.$$

$$f_y = \frac{2y}{x^2+y^2}, \quad f_{yy} = \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2}$$

$$\Rightarrow \Delta f = \frac{4}{x^2+y^2} - \frac{4(x^2+y^2)}{(x^2+y^2)^2} = 0.$$

So $\oint_{K \sqcup C_\varepsilon} \nabla f \cdot \vec{n}\, ds = 0$. On the other hand,

$$\oint_{K \sqcup C_\varepsilon} \nabla f \cdot \vec{n}\, ds = \oint_K \nabla f \cdot \vec{n}\, ds + \int_{C_\varepsilon} \nabla f \cdot \vec{n}\, ds$$

$$= \oint \nabla f \cdot \vec{n}\, ds - \int_{C_\varepsilon^-} \nabla f \cdot \vec{n}\, ds$$

C_ε^- ↳ counter-clockwise

Hence $\oint_K \nabla f \cdot \vec{n}\, ds = \int_{C_\varepsilon^-} \nabla f \cdot \vec{n}\, ds$

$$= \int_{C_\varepsilon^-} \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}\right) \cdot \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right) ds = \int_0^{2\pi} 2 \, d\theta = 4\pi.$$

Case 2: $(0,0) \notin R$, then $\nabla f \in C^1(\overline{R})$, so

$$\oint_K \nabla f \cdot \vec{n}\, ds = \int_R \Delta f \, dA = 0.$$

▨